M.R.P. 1003
S.C.11/217
17th September, 1956.

A non-geostrophic extension of the Sawyer-Bushby model of the atmosphere suitable for numerical integration

by

E. Knighting.

(Meteorological Office)

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

At the time of issue this report has not been published. Any author desiring to quote this paper should refer to it as "A paper of the Meteorological Research Committee (London).
M.R.P. No.1003 by Mr. E. Knighting
A copy is available in the Library of the Meteorological Office."

M.R.P. 1003
S.C.II/217
17th September, 1956

AIR MINISTRY
METEOROLOGICAL RESEARCH COMMITTEE

A non-geostrophic extension of the Sawyer-Bushby model of the atmosphere suitable for numerical integration

by

E. Knighting (Meteorological Office)

1. Introduction

The atmospheric models that have been used for the numerical prediction of the pressure field have used the quasi-geostrophic assumption, that the velocity and vorticity can be obtained from the geopotential field. The success obtained in forecasting at 500 mb with all such models has been rather similar and is characterised by correlation coefficients somewhat similar to those obtained by using the orthodox forecasting methods. Means have been sought to relax the quasi-geostrophic assumption and have led several people (see Bushby (1956)) to examine the effect of using a stream function to define the velocity and relating it to the geopotential by the ordinary equations of motion. In effect one has to obtain the stream function from the geopotential using the balance equation, carry forward the time integrations as usual and finally recover the geopotential from the forecast stream function. The motion is, of course, non-divergent and it is the purpose of this note to examine the effect of introducing a divergent component of the velocity in accordance with Helmholtz's theorem for a plane vector. An extensive treatment of this problem has recently been given by Thompson (1956).

In keeping with the general practice at the Meteorological Office, Dunstable, of examining the simplest models first the equations have been written down for the simplest two-level model in which the divergence near the surface is not taken to be zero, while that at 500 mb is taken as zero. The method of solution is rather different from that normally used, in that the vertical velocity is first determined as the key to the development.

A simplification, which is made in section 4, reduces the difficulty of solving the equations at the expense of making a physical assumption which is not easy to understand and which involves rather high derivatives. The implication of the physical assumption is clear and shows that we have retained the quasi-geostrophic assumption in calculating the local rates of change of thermal vorticity. Among the other alternative simplifications, we could take as zero the Jacobian on the left hand side of equation (14), taking the stream function as the geopotential at 1000 mb. These simplifications should be explored in turn when numerical experiments are made using this model.

2. The assumptions

The model is bounded by two constant pressure surfaces p_0 and p_1, which will be taken for practical calculations as 1000 and 200 mb. The vertical velocity dp/dt has a parabolic distribution with height and vanishes at p_0 and p_1. The thermal wind is unidirectional and is linear with pressure. Writing suffixes zero and one to refer to the levels p_0 and p_1, and using the vertical co-ordinate

$$\alpha = \frac{p_0 + p_1 - 2p}{p_0 - p_1} \qquad (1)$$

with suffix m referring to values at $\alpha = 0$, and with V' the thermal wind between $\alpha = -1$ and $\alpha = 0$ (or $\alpha = 0$, $\alpha = +1$) the following equations are direct consequences of the assumptions

$$V = V_m + \alpha V' \tag{2}$$

$$\zeta = \zeta_m + \alpha \zeta' \tag{3}$$

$$\mathrm{div}\, V = \mathrm{div}\, V_m + \alpha\, \mathrm{div}\, V' \tag{4}$$

$$\frac{dp}{dt} = \omega = \omega_m (1 - \alpha^2) \tag{5}$$

where V is the wind corresponding to the vertical co-ordinate α and ζ, ζ_m and ζ' are the vorticities corresponding to the velocities V, V_m and V'.

3. Prognostic equations for ζ_m, ζ' and h'

Integrating the vorticity equation between $\alpha = \pm 1$, neglecting the twisting term, gives

$$\frac{\partial \zeta_m}{\partial t} + V_m . \nabla (\zeta_m + f) + \tfrac{1}{3} V' . \nabla \zeta' = \frac{8 \omega_m \zeta'}{3(p_0 - p_1)} \tag{6}$$

The difference between the vorticity equation at $\alpha = \pm 1$ gives

$$\frac{\partial \zeta'}{\partial t} + V_m . \nabla \zeta' + V' . \nabla (\zeta_m + f) = \tfrac{1}{2} \left[(\zeta_m + f) \left\{ \left(\frac{\partial \omega}{\partial p} \right)_1 - \left(\frac{\partial \omega}{\partial p} \right)_0 \right\} + \zeta' \left\{ \left(\frac{\partial \omega}{\partial p} \right)_1 + \left(\frac{\partial \omega}{\partial p} \right)_0 \right\} \right] \tag{7}$$

and using equation (5) gives

$$\frac{\partial \zeta'}{\partial t} + V_m . \nabla \zeta' + V' . \nabla (\zeta_m + f) = \frac{4 \omega_m}{p_0 - p_1} (\zeta_m + f) \tag{8}$$

If ϕ is the height of a pressure surface, ϕ' the thickness between the surfaces $\alpha = -1$, 0, the adiabatic equation gives

$$\int_{-1}^{0} \left(\frac{\partial}{\partial t} + V . \nabla \right) \frac{\partial \phi}{\partial \alpha} \, d\alpha + \frac{2}{p_0 - p_1} \int_{-1}^{0} \omega \frac{\partial \phi}{\partial p} \frac{\partial \log \theta}{\partial p} \, d\alpha = 0 \tag{9}$$

In the second integral we use equation (5) to replace ω and take values of $\partial \phi / \partial p$ and $\partial \log \theta / \partial p$ which are given by some standard atmosphere. The

first integral is

$$\int_{-1}^{0} \left(\frac{\partial}{\partial t} + V_m \cdot \nabla \right) \frac{\partial \phi}{\partial \alpha} d\alpha + \int_{-1}^{0} V' \cdot \nabla \left(\alpha \frac{\partial \phi}{\partial \alpha} \right) d\alpha$$

and we must approximate to the second integral. For example we may take $\phi = \phi_m + \phi' \alpha$, which yields

$$\frac{\partial \phi'}{\partial t} + V_m \cdot \nabla \phi' - \tfrac{1}{2} V' \cdot \nabla \phi' = K \omega_m \qquad (10)$$

where K is determined by the standard atmosphere.

4. Introduction of stream and potential function

We now have three equations (6), (8) and (10) with the unknowns $\zeta_m, V_m, \zeta', V', \phi', \omega_m$. Making the geostrophic assumption relates both ζ_m and V_m with ϕ_m, ζ' and V' with ϕ', leaving three equations and three unknowns. More generally we must write

$$V = K \wedge \nabla \psi + \nabla \chi \qquad (11)$$

and substituting these expressions in equations (6), (8) and (10) leads to variables $\psi_m, \chi_m, \psi', \chi', \phi', \omega_m$. The original data will give ϕ' and ϕ_m. In this simple model we have

$$\nabla^2 \chi_m = -\left(\frac{\partial \omega}{\partial p} \right)_m = 0 \qquad (12a)$$

$$\nabla^2 \chi' - \left(\frac{\partial \omega}{\partial p} \right)_0 = -\frac{4 \omega_m}{p_0 - p_1} \qquad (12b)$$

so that χ_m may be taken as zero at all times. Using a suffix zero to represent values at $\alpha = -1$, i.e. at p_0 then $V_0 = V_m - V'$ and $\psi_0 = \psi_m - \psi'$ and the balance equation suffices to give ψ_m and ψ_0 originally,

$$2J\left(\frac{\partial \psi_m}{\partial x}, \frac{\partial \psi_m}{\partial y} \right) + f \nabla^2 \psi_m = \nabla^2 \phi_m \qquad (13)$$

$$2J\left(\frac{\partial \psi_0}{\partial x}, \frac{\partial \psi_0}{\partial y} \right) + f \nabla^2 \psi_0 = \nabla^2 \phi_0 \qquad (14)$$

where J denotes the usual Jacobian.

The predictive equations are now

$$\frac{\partial}{\partial t} \nabla^2 \psi_m + J\left(\psi_m, \nabla^2 \psi_m + f \right) + \tfrac{1}{3} J_1\left(\psi', \chi', \nabla^2 \psi' \right) = \frac{8 \omega_m \nabla^2 \psi'}{3(p_0 - p_1)} \qquad (15)$$

$$\frac{\partial}{\partial t}\nabla^2\psi' + J(\psi_m, \nabla^2\psi') + J_1(\psi', \chi', \nabla^2\psi_m + f) = \frac{4\omega_m \nabla^2(\psi_m) + f}{p_0 - p_1} \qquad (16)$$

$$\frac{\partial \phi'}{\partial t} + J(\psi_m, \phi') - \tfrac{1}{2} J_1(\psi', \chi', \phi') = K\omega_m \qquad (17)$$

where $$J_1(\psi, \chi, \lambda) = \frac{\partial \psi}{\partial x}\frac{\partial \lambda}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \lambda}{\partial x} + \frac{\partial \chi}{\partial x}\frac{\partial \lambda}{\partial x} + \frac{\partial \chi}{\partial y}\frac{\partial \lambda}{\partial y} \qquad (18)$$

Equations (12b) to (17) are six equations with unknowns $\psi_m, \psi_0, \omega_m, \phi_m, \phi_0, \chi_0$ and form a closed set.

5. Some simplifications

The equations have already been simplified by the omission in equation (14) of the terms in $\partial\nabla^2\chi_0/\partial t$ and $\nabla^2\chi_0$ which could have been retained without affecting the number of equations and variables. Even with this simplification the equations are difficult to solve. The suggestion is that

$$\frac{\partial}{\partial t} J\left(\frac{\partial \phi_m}{\partial x}, \frac{\partial \psi_m}{\partial y}\right) \simeq \frac{\partial}{\partial t} J\left(\frac{\partial \phi_0}{\partial x}, \frac{\partial \phi_0}{\partial y}\right) \qquad (19)$$

which is true in areas where the geostrophic approximation is fairly good and implies a linkage in the curvature of the ϕ_m and ϕ_0 surfaces. From (13) and (14) we have

$$\nabla^2 \frac{\partial \phi'}{\partial t} = f \nabla^2 \frac{\partial \psi'}{\partial t}$$

which implies that the local rate of change of thermal vorticity can be calculated from the thickness pattern, as is done in geostrophic motion. Then taking ∇^2 of equation (17) and subtracting (16) we have

$$K\nabla^2\omega_m + \omega_m \nabla^2 K + 2\nabla\omega_m \cdot \nabla K - \frac{4\nabla^2\psi_m + f}{p_0 - p_1} = F \qquad (20)$$

where F is a known function of x, y. We solve equation (20) as a boundary value problem, and hence (12b), (15), (16) and (17), a series of Poisson equations except for the very simple (17).

References

(1) Bushby, F. H. and Huckle, V. M. 1956 Meteorological Research Papers, Air Ministry, London, No. 956.

(2) Thompson, P. D. 1956 J. Met., 13, p. 251.